

# Coordinate Transformations for World War 1 Topographic Maps

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## Abstract

Topographic maps (1:40,000) used by the British Army on the Western Front in World War 1 had a five-part Grid Reference System consisting of: (1) **Map Number**; (2) **Letter-Square** – 24 letter-squares A to X on each map; (3) **Number-Square** – 36 (and sometimes 30) 1000-yard squares in each letter square; (4) **Minor-Square** – four 500-yard squares denoted a,b,c,d in each number-square; (5) **Small-Square** –  $10 \times 10 = 100$  small-squares in a minor-square. Letter and number grid references (e.g. Woesten<sup>1</sup> X: 28.A.6.b.73) cannot be used by modern GPS navigation devices that require geographical coordinates (latitude and longitude) or current map grid coordinates. This paper demonstrates a method of transforming WW1 grid references to Universal Transverse Mercator (UTM) grid coordinates using Google Maps to obtain geographical coordinates, Geographic to UTM grid conversion and a 2D Conformal transformation.

## Introduction

At the beginning of World War 1 (WW1) the British Army was supplied with Ordnance Survey medium scale (1:100,000 and 1:80,000) topographic maps deemed adequate for the expected war of movement. But after the opening battles of Flanders in western Belgium the front stabilized into entrenched positions extending from the coast of Belgium to the Swiss border and the German and Allied armies engaged in a war of attrition. The entrenchment of ground forces led to a change in the usual artillery dispositions and battery commanders required more detailed large scale maps. From early 1915 these new maps were of three scales: 1:40,000 (designated as series GSGS 2743 by the Geographic Section, General Staff); 1:20,000 (GSGS 2742); and 1:10,000 (GSGS 3062).

The WW1 British Army Grid Reference System was based on ‘squaring’ where a map was successively divided into smaller and smaller grid squares (sometimes rectangles); each square having an identifier (letter or number) unique to that map. Understanding the dimensions of the squares (which were in yards) and then parsing the grid reference into its constituent parts enables a point X to be located by two distances, east and north of the south-west (SW) corner of the map grid. If this SW corner is the origin of a coordinate system then the distances become the east (E) and north (N) coordinates of X. This process of converting a Grid References to Grid Coordinates could be ‘programmed’ either by using an appropriate computer language (e.g. C, Java, Python, etc.) or a spreadsheet (e.g. Microsoft Excel<sup>®</sup>).

Many topographic features (e.g. roads, railways, rivers, road intersections, bridges, etc.) appearing on WW1 maps are still existing today and visible on Google Maps (<https://maps.google.com/>) and geographical coordinates (latitude  $\phi$  and longitude  $\lambda$ ) can be determined (move the cursor  to the point; left-click;  $\phi, \lambda$  in decimal degrees appear in a screen text-box). Assuming  $\phi, \lambda$  are related to the World Geodetic System 1984 (WGS84) ellipsoid<sup>2</sup>, geographic coordinates can be converted to Universal Transverse Mercator (UTM) grid coordinates E,N and UTM Zone no. using readily available software (e.g. Convert Lat Long to UTM <http://www.latlong.net/>). As an example the latitude and longitude of X at Woesten are 50.9007 degrees North and 2.7891 degrees East respectively are converted to UTM Zone 31 grid coordinates  $E = 485169.67$  m and  $N = 5638803.63$  m. Appendix A has an explanation of the UTM projection and the UTM grid.

So, it is possible to locate a feature on a 1:40,000 WW1 topographic map by a Grid Reference and convert this to a Grid Coordinate (in yards) related to the SW corner of the map sheet (e.g. Woesten X: 28.A.6.b.73 = 5850 E, 21650 N). And, locating this same feature on Google Maps, determine its geographical coordinates  $\phi, \lambda$  and then convert them to UTM Zone 31 grid coordinates.

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<sup>1</sup> Woesten is a rural Belgian village in the west Flanders province about 5 km NW of Ypres. The point X on the 1:40,000 Map No. 28 (Grid Reference: 28.A.6.b.73) appears as the intersection of the N8 road and Steenstraat on Google Maps and the latitude and longitude are 50.9007 degrees North and 2.7891 degrees East respectively.

<sup>2</sup> An ellipsoid is a three dimensional surface of revolution created by rotating an ellipse ( $a, b$ ) about its minor axis where  $a$  is the semi-major axis and  $b$  is the semi-minor axis and  $a > b$ . A reference ellipsoid is used as a mathematical approximation of the earth and is usually defined by specifying the semi major axis  $a$  and the flattening  $f = (a - b)/a$ . Various reference ellipsoids are used in mapping applications and one such ellipsoid is the reference ellipsoid of the World Geodetic System 1984 (WGS84) defined by  $a = 6378137$  m and flattening  $f = 1/298.257223563$ .

Using a selection of such ‘control’ points across the map, grid coordinates in two different rectangular systems can be obtained – (i) British Army grid coordinates and (ii) UTM Zone 31 grid coordinates – and these values used to determine the parameters of a 2D Conformal transformation.

A 2D Conformal transformation has the general property of preserving angles at a point and the shape of small regions and mathematically links  $E, N$  rectangular coordinates of one system to  $U, V$  rectangular coordinates of another system via a matrix equation of the form

$$\begin{bmatrix} E_k \\ N_k \end{bmatrix} = s \mathbf{R}_\theta \begin{bmatrix} U_k \\ V_k \end{bmatrix} + \begin{bmatrix} t_E \\ t_N \end{bmatrix} \quad (1)$$

where  $\begin{bmatrix} E_k \\ N_k \end{bmatrix}$  and  $\begin{bmatrix} U_k \\ V_k \end{bmatrix}$  are vectors containing the coordinates of the  $k$ th control point,  $s$  is a scale factor,  $\begin{bmatrix} t_E \\ t_N \end{bmatrix}$  is a

vector of translations between the coordinate origins and  $\mathbf{R}_\theta = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$  is a rotation matrix.  $\theta$  is a rotation

angle (measured positive anti-clockwise from the  $X$ -axis). In words, equation (1) can be stated as:  $U, V$  coordinates are rotated; scaled; and then translated to  $E, N$  coordinates and the four parameters of the transformation are scale  $s$ , rotation  $\theta$  and translations  $t_E, t_N$ . A minimum of two control points is required to determine the parameters, but it is good practice to have four or more and use a mathematical estimation process to obtain the most probable values of the parameters. Appendix B has an explanation of a 2D conformal transformation and a method of solution for the parameters.

Once the transformation parameters (scale, rotation and translations) have been obtained for a particular map, then the British Army grid coordinates of any point can be transformed to UTM Zone 31 coordinates.

This paper gives a detailed explanation of the British Army Grid Reference System and the conversion of these letter/number references to Grid coordinates related to the SW corner of the 1:40,000 map sheet. A worked example of a 2D conformal transformation using selected control points is also provided.

## British Army Grid Reference System for 1:40,000 Topographic Maps

To formalize position descriptions all British series maps were given unique numbers (and sometimes names) based on the Belgium (metric) mapping grid system aligned with the meridian through Brussels and dividing the country into 71 blocks measuring 32,000 metres (east-west) by 20,000 metres (north-south). Each block in this grid constituted a single map sheet at the 1:40,000 scale with sheet corners carrying distances in metres from Brussels. Each block was designated by a unique **Map Number** from 1 to 71 and blocks in a row that extended westwards into France were distinguished by a letter suffix. For example, the blocks extending into France west of the Belgium block 62 were numbered 62A, 62B, 62C and 62D. These numbers became the reference numbers for map sheets in the 1:40,000 series (Murray 1988).

The 1:40,000 series maps were overlaid with a grid that divided each sheet into 24 **Letter-Squares**, four rows of six squares each (4 rows  $\times$  6 columns) denoted by upper-case letters A to X. The squares in the inner rows (row 2: G to L, and row 3: M to R) measured 6,000 yards square; but the squares in the outer rows (row 1: A to F, and row 4: S to X) were in fact rectangles measuring 5,000 yards north-south by 6,000 yards east-west. The grid was placed with its central point in the middle of the 1:40,000 sheet and was allowed to overlap the sheet boundaries which were metric grid lines. Letter-squares were subdivided into 1,000-yard **Number-Squares**; 36 (6 rows  $\times$  6 columns) in the larger letter squares and 30 (5 rows  $\times$  6 columns) in the smaller letter squares. Each of the 1,000-yard number-squares were then quartered into **Minor-Squares** of 500-yard sides and denoted by lower-case letters a to d. Minor-squares were further divided into  $10 \times 10 = 100$  **Small-Squares** of 50-yard sides.

A point of interest within a minor-square would be located by two distances in units of tenths of the minor-square boundaries (1 tenth = 50 yards). The first being a distance east from the western boundary of the minor-square and the second being a north distance measured from the southern boundary. If more accuracy was required then the distances would be expressed in units of hundredths (1 hundredth = 5 yards).

For example a Grid Reference for point X at Woesten is: 28.A.6.b.73

where     **28** is the Map Sheet Number,  
          **A** is the Letter-Square,  
          **6** is the Number-Square,  
          **b** is the Minor-Square.

**73** is a 2-digit number of parts 7 and 3. If the boundaries of minor-square 'b' are divided into tenths then *X* is a distance 7 tenths east of the western boundary and 3 tenths north of the southern boundary of the minor square. The east distance is written before the north distance and 1 tenth = 50 yards. This reference could also be written as 28.A.6.b.7030 where the four digits 7030 represent 70 hundredths east and 30 hundredths north where 1 hundredth = 5 yards.

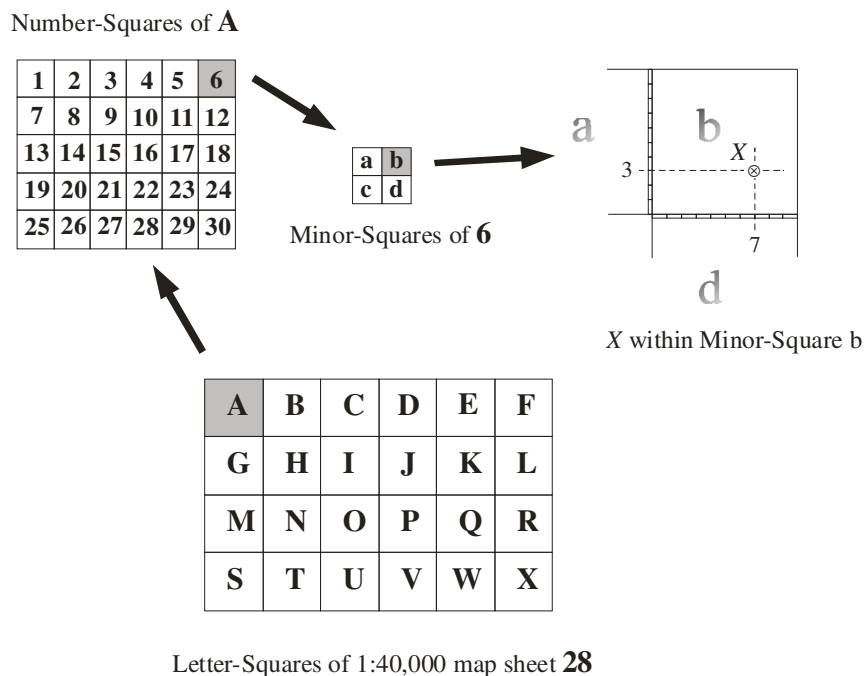


Figure 1. Diagrammatic view of Grid Reference Woesten *X*: 28.A.6.b.73



Figure 2. Woesten *X*: 28.A.6.b.73 (road intersection)

Part of Map No. 28 *Belgium and part of France*, edition 4, series GSGS 2743, date 1918/01, scale 1:40,000

McMaster University Library, McMaster University, Ontario, Canada

<http://digitalarchive.mcmaster.ca/islandora/object/macrepo%3A4193/-/collection>

Note on Figure 2 that the east-west gridline that is the north boundary of Number-square 6 is beyond the boundary of the map. This is a consequence of a yards-grid overlaying a metric map sheet.

## Division of the 1:40,000 Series into 1:20,000 and 1:10,000 maps

The 1:40,000 topographic maps were also divided into quadrants denoted NW, NE, SW and SE and maps of these quadrants produced at scales of 1:20,000. Quadrants were subdivided into four sections numbered 1, 2, 3 and 4 and denoted NW1, SE3, NE4, etc. and maps of these sections produced at scales of 1:10,000.

1:20,000 sheet 28NW		28NE	
Sheet 28 1:40,000			
1:10,000 sheet 28SW1	28SW2		
28SW		28SE	
28SW3	28SW4		

Figure 3. Division of 1:40,000 map 28 into four 1:20,000 maps and sixteen 1:10,000 maps

The road intersection Woesten X: 28.A.6.b.73 on the 1:40,000 map 28 (see Figure 3) is also on the 1:20,000 map 28NW (see Figure 4) and the 1:10,000 map 28NW2 (see Figure 5)

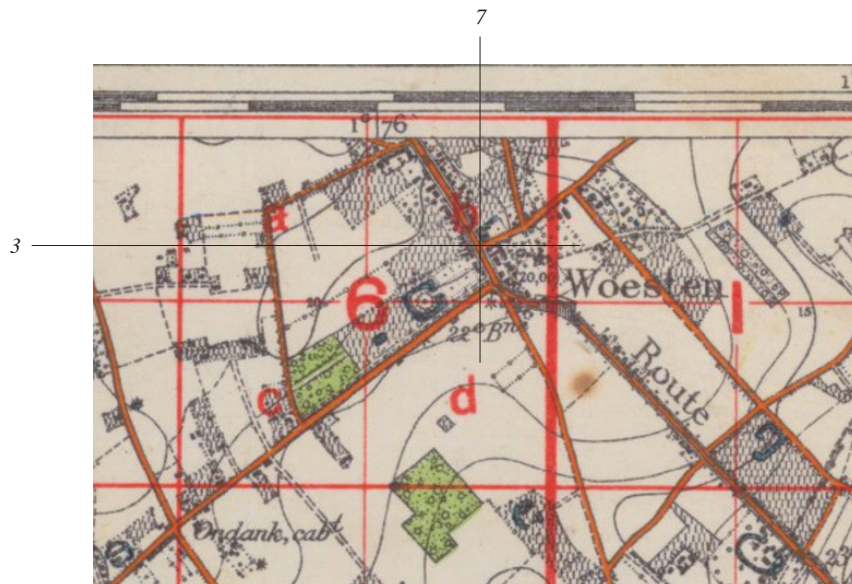


Figure 4. Woesten X: 28NW.A.6.b.73 (road intersection)

Part of Map No. 28NW *Poperinghe-Ypres*, series GSGS 2742, scale 1:20,000  
 McMaster University Library, McMaster University, Ontario, Canada  
<http://digitalarchive.mcmaster.ca/islandora/object/macrepo%3A4052/-/collection>

Note on Figure 4 that the east-west gridline that is the north boundary of Number-squares A.6 and B.1 is beyond the boundary of the map. This is a consequence of a yards-grid overlaying a metric map sheet.

Figure 5 shows point X at Woesten on a 1:10,000 map and at this scale features within Minor-Squares were often located by dividing the boundaries into hundredths (1 hundredth = 5 yards) leading to four-digit grid coordinates. For example point X (road intersection) is now described as 28NW2.A.6.b.6628. The last four digits 6628 represent 66 hundredths east and 28 hundredths north (or 6.6 tenths and 2.8 tenths).



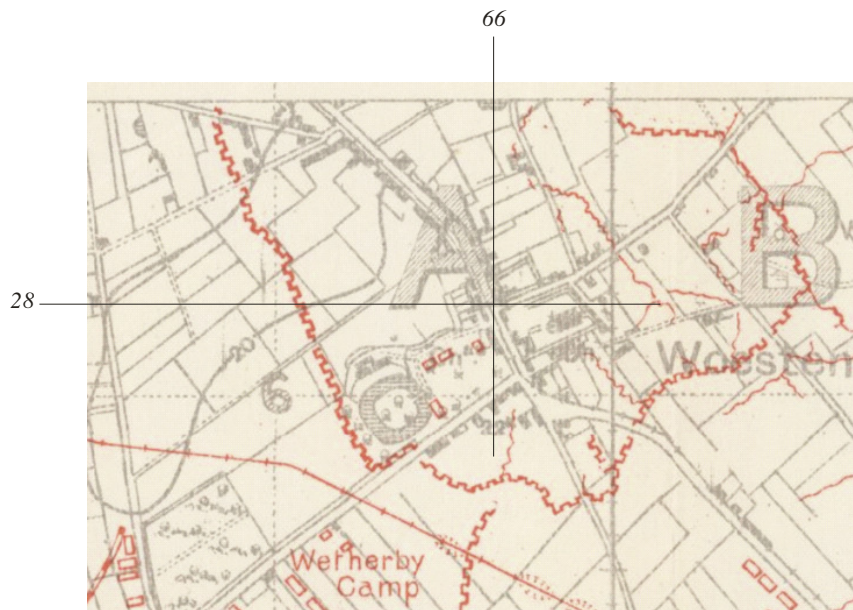


Figure 5. Woesten X: 28NW2.A.6.b.6628 (road intersection)

Part of Map No. 28NW1 *Elverdinghe*, edition 4a, series GSGS 3062, scale 1:10,000  
 McMaster University Library, McMaster University, Ontario, Canada  
<http://digitalarchive.mcmaster.ca/islandora/object/macrepo%3A66680/-/collection>

## British Army Grid Coordinates

A grid reference for a point on a 1:40,000 map (e.g. Woesten X: 28.A.6.b.73) can be turned into a east and north coordinates related to an origin at the SW corner of the grid, which is the SW corner of Letter-Square S. Referring to Figure 1 the coordinates of: (i) the SW corner of Letter-Square A; (ii) the SW corner of Number-Square 6; (iii) the SW corner of Minor-Square b; and finally (iv) the point of X are obtained as follows

- (i) The SW corner of Letter-Square A is 17,000 yards north of the origin and its coordinates are 0 E, 17000 N.  
 Remember here that the Letter-Squares in the 1st and 4th rows (rows A to F and S to X) are in fact rectangles with north-south sides of 5000 yards and east-west sides of 6000 yards and contain 30 (5 rows  $\times$  6 columns) of Number-Squares.
- (ii) The SW corner of Number-Square 6 is 4,000 yards north and 5,000 yards east of the SW corner of Letter-Square A, so the coordinates of the SW corner of Number-Square 6 are  $0 + 5000 = 5000$  E,  $17000 + 4000 = 21000$  N.
- (iii) The SW corner of Minor-Square b is 500 yards north and 500 yards east of the SW corner of Number-Square 6, so the coordinates of the SW corner of Minor-Square b are  $5000 + 500 = 5500$  E,  $21000 + 500 = 21500$  N.
- (iv) The coordinates of X are now obtained by taking the last part of the Grid Reference (73) and separating the digits (7 and 3) that are in units of tenths. So X is 7 tenths east and 3 tenths north of the SW corner of the Minor-Square and since 1 tenth = 50 yards then the coordinates of X are  $5500 + (7 \times 50) = 5850$  E,  $21500 + (3 \times 50) = 21650$  N.

## Google Maps, Geographical Coordinates and UTM Grid Coordinates

For the purposes of determining the parameters of a 2D Conformal transformation a number of control points covering the region of interest are required. For this exercise, control points are topographic features appearing on the WW1 topographic Map No. 28 (1:40,000) and still obvious today on Google Maps. For the example that follows the control points are road intersections.

Using Google Maps (<https://maps.google.com/>) and typing a village or town name and country into the search box brings up a number of options. Selecting the appropriate location displays the town at the centre of the map and the user has the option of toggling between Map and Earth (satellite image) views. Using the zoom feature, the user can

locate the topographic feature that appears on the WW1 map. For example, the village of Woesten is shown on Figure 6



Figure 6. Google Maps image of Woesten, Belgium

Zooming in shows the intersection of the N8 road and Steenstraat which appears on the 1:40,000 topographic map (see Figure 2) that is Woesten X: 28.A.6.b.73 and a left-click on the mouse displays the latitude and longitude in decimal degrees.

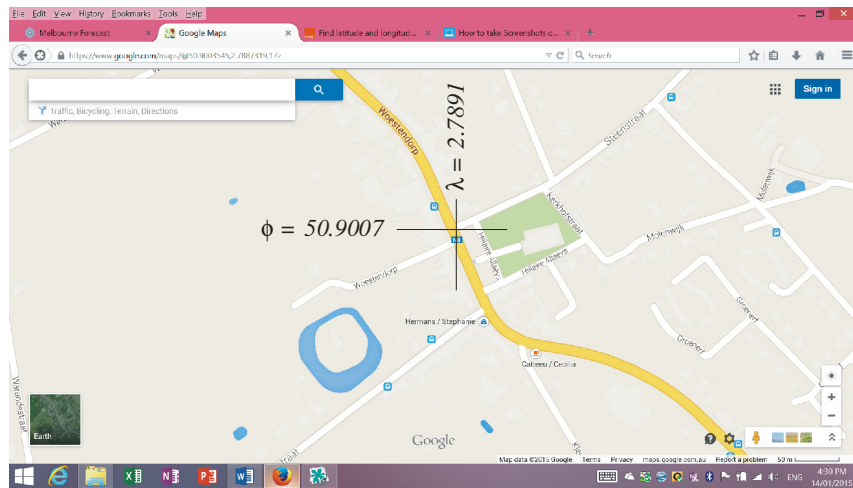


Figure 7: Woesten X, the intersection of the N8 road and Steenstraat at latitude 50.9007 degrees north and longitude 2.7891 degrees east.

Latitude and longitude are geographical coordinates used to locate points on the earth's reference ellipsoid. The latitude of a point  $P$  is the angular measure along a meridian<sup>3</sup> from the equator of the ellipsoid to  $P$ . Latitudes are measured from  $0^\circ$  to  $\pm 90^\circ$  (positive north, negative south of the equator). The longitude of  $P$  is the angular measure along the equator between the Greenwich meridian plane (the reference meridian) and the meridian plane through  $P$ . Longitudes are measured  $0^\circ$  to  $\pm 180^\circ$  (positive east, negative west of Greenwich). A parallel of latitude is a line on the ellipsoid along which the latitude is constant and is created by intersecting the ellipsoid with a plane perpendicular to the north-south axis of the ellipsoid. The equator is a parallel of latitude.

$\phi, \lambda$  are displayed on Google Maps in decimal degrees to 6 decimal digits. Consider the last digit (0.000001) as representing small changes  $\delta\phi = \delta\lambda = 0.000001$  degrees ; then the corresponding small distances  $\delta s_m, \delta s_p$  along the

<sup>3</sup> A plane that intersects the ellipsoid and contains the north and south poles is a meridian plane (or meridian section) and the line of intersection on the surface of the ellipsoid is a meridian.

meridian of longitude (north-south) and parallel of latitude (east-west) respectively on the WGS84 ellipsoid (earth reference surface) are given by

$$\delta s_m = r_m \delta \phi \quad \text{and} \quad \delta s_p = r_p \cos \phi \delta \lambda \quad (2)$$

where  $r_m$ ,  $r_p$  are respectively radii of curvature<sup>4</sup> of the ellipsoid in the meridian plane and the prime vertical plane. These formula can be simplified by using the mean radius of curvature  $R_M$  where

$$R_M = \sqrt{r_m r_p} = \frac{a(1-f)}{1-f(2-f)\sin^2 \phi} \quad (3)$$

and  $a$  and  $f$  are respectively; the semi-major axis length of the ellipsoid and the flattening of the ellipsoid. For the WGS84 ellipsoid  $a = 6378137$  m and  $f = 1/298.257223563$ .

With  $r_m \approx r_p = R_M$  equations (2) become

$$\delta s_m \approx R_M \delta \phi \quad \text{and} \quad \delta s_p \approx R_M \cos \phi \delta \lambda \quad (4)$$

For  $\phi = 50^\circ$   $R_M = 6381822.82$  m and using (4) with  $\delta \phi = \delta \lambda = 0.000001$  degrees (but expressed in radians<sup>5</sup>)  $\delta s_m = 0.11$  m  $\delta s_p = 0.07$  m. This is an unrealistic impression of the accuracy of the map (or satellite image) shown in Google Maps.

A more realistic expression of accuracy would be to round values of latitude and longitude to 4 decimal places. This implies a rounding error of 0.00005 degrees; and for  $\phi = 50^\circ$   $R_M = 6381822.82$  m the rounding error corresponds to  $\delta s_m = 5.57$  m,  $\delta s_p = 3.58$  m (distances along the meridian and parallel respectively)

Maps (which include digital images) are mathematical transformations of points on the ellipsoid (the earth's reference surface) to a plane. We call these transformations 'map projections' and there are several hundred to select from. For topographic mapping, conformal map projections are usually chosen and conformal projections have the useful property that the scale factor<sup>6</sup> in any direction about a point on the projection is constant and angles at a point between other points on the projection are the same as corresponding angles on the ellipsoid. These two properties mean that the shape of small regions on the projection is the same as the corresponding shape on the ellipsoid; but this property does not extend to large regions of the ellipsoid. Amongst the many conformal projections the Transverse Mercator (TM) is useful for regions of the ellipsoid having small extent in longitude but large extent in latitude and lends itself to mapping the ellipsoid in a series of longitude zones.

For example, Figure 8 shows a Transverse Mercator projection of part of the earth between longitudes  $60^\circ$  and  $180^\circ$  as three zones, each  $40^\circ$  wide and touching at the equator. The meridians and parallels in each zone form an orthogonal network of lines known as the graticule and in each zone there are only two straight lines; the equator and the central meridian  $\lambda_0$ ; all other lines are complex curves. These two straight lines (equator and central meridian) are the basis of a grid that can be used to locate points on the projection in each zone; thus we have the TM Graticule and the TM Grid of a zone of the earth.

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<sup>4</sup> At a point on an ellipsoid there are two sections (or planes) of interest: the meridian plane containing the North and South poles and the prime vertical plane which is perpendicular to the meridian plane. These two sections of the ellipsoid are ellipses (both of different dimensions) and a point on an ellipse has a radius of curvature that changes as the point moves along the ellipse (a point on a circle has radius that remains constant as the point moves along the circle). It turns out that the radii of curvature in the meridian plane and the prime vertical plane are the smallest and the greatest respectively of all the possible radii of sections of an ellipsoid at a particular point. The equations for the radii are complicated and beyond the scope of this paper. Interested readers can consult geodesy texts.

<sup>5</sup> radians are an alternative angular measure to degrees. There are  $2\pi$  radians =  $360^\circ$  in a revolution so the conversion is: 1 radian =  $180/\pi$  degrees = 57.2957795131 degrees.

<sup>6</sup> scale factor  $m$  is a ratio and  $m = ds/dS$  where  $ds$  is a differentially small distance on the earth's surface and  $dS$  is a differentially small distance on the map projection.

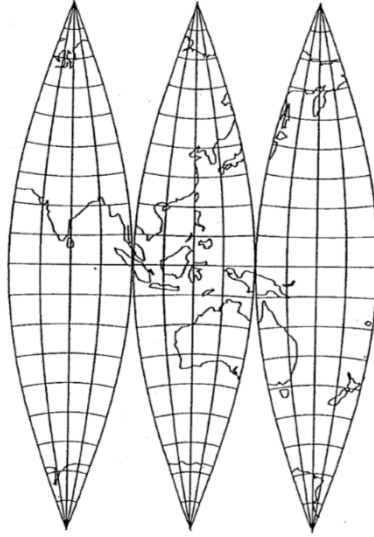


Figure 8. Transverse Mercator projection of part of the earth.

Graticule interval is  $10^\circ$  and each zone is  $40^\circ$  wide with central meridians  $\lambda_0 = 80^\circ$ ,  $\lambda_0 = 120^\circ$  and  $\lambda_0 = 160^\circ$ .

The scale factor along the central meridian of a TM zone is constant but at every other point on the projection it varies and increases to maximum values along zone boundaries. To restrict the range of scale factors (and hence limit distortions) the zone widths can be decreased and a central meridian scale factor  $m_0$  imposed. The Universal Transverse Mercator (UTM) is a particular case of the general Transverse Mercator projection and is used to map the ellipsoid in a series of 60 zones each  $6^\circ$  wide in longitude. UTM Zone 1 has a central meridian  $-177^\circ$  and zones are numbered eastwards with zones 30 and 31 having central meridians  $\lambda_0 = -3^\circ$  and  $\lambda_0 = +3^\circ$  respectively and Zone 60 has central meridian  $\lambda_0 = 177^\circ$ . The central meridian of any UTM Zone is given by

$$\lambda_0 = \text{Zone} \times 6^\circ - 183^\circ \quad (5)$$

And the UTM Zone number within which a point of longitude  $\lambda$  lies is given by

$$\text{Zone} = \text{round} \left( \frac{\lambda + 183}{6} \right) \quad (6)$$

where the *round*( ) function rounds to the nearest integer.

The central meridian scale factor of a UTM zone is  $m_0 = 0.9996$ .

Every UTM zone has a northern and a southern part and has a grid superimposed over the graticule that is aligned with the central meridian and the equator. The intersection of the central meridian and the equator is the true origin of a zone and for coordination purposes each zone has two false origins for East and North UTM Grid coordinates. For the northern part of a zone the false origin lies on the equator, 500,000 m west of the true origin. For the southern part of a zone the false origin is 500,000 west and 10,000,000 m south of the true origin. Appendix A has additional information about the UTM projection and the UTM Grid.

The equations for the transformations  $\phi, \lambda \Leftrightarrow E, N$  for the Transverse Mercator projection were originally developed in 1822 by the mathematician C.F. Gauss (1777–1855) and L. Krueger published extensive studies including suitable formulae for computation on the projection in 1922. A recent publication by Karney (2011) gives a detailed modern development of formulae that give nanometre precision for points within 3000 km of the central meridian of a zone, and these formulae (in a modified form) have been recently adopted by the U.S. National Geospatial-Intelligence Agency (NGA 2014). A summary of Karney's formula (in a truncated form) can be found on the Internet at [http://en.wikipedia.org/wiki/Universal\\_Transverse\\_Mercator\\_coordinate\\_system](http://en.wikipedia.org/wiki/Universal_Transverse_Mercator_coordinate_system). These modern developments of formulae for the transformations  $\phi, \lambda \Leftrightarrow E, N$  supersede older sets of equations (e.g. Redfearn's equations in Australia) but the user of software is generally unaware of the source of equations and simply assumes that they are 'fit for purpose'.



Latitude and longitude  $(\phi, \lambda)$  can be converted to UTM Grid coordinates  $(E, N)$  using software available on the Internet. For example, Convert Lat Long to UTM at <http://www.latlong.net/> converts  $\phi, \lambda$  to  $E, N$  and also shows the location on Google Maps. Unfortunately it does not offer the reverse conversion, Grid to Geographic  $(E, N$  to  $\phi, \lambda)$ , but other sites (e.g. Geoscience Australia at <http://www.ga.gov.au/>) offer Grid to Geographic conversions.

The effects of rounding  $\phi, \lambda$  to the nearest 0.0001 degrees (which induces a possible rounding error of 0.00005 degrees) on the calculation of  $E, N$  coordinates is a complex problem but can be expressed mathematically by the matrix equation (Deakin & Hunter 2015)

$$\begin{bmatrix} \delta E \\ \delta N \end{bmatrix} \approx \frac{R_M}{1 - \cos^2 \phi \sin^2 \omega} \begin{bmatrix} -\sin \phi \sin \omega & \cos \phi \cos \omega \\ \cos \omega & \sin \phi \cos \phi \sin \omega \end{bmatrix} \begin{bmatrix} \delta \phi \\ \delta \lambda \end{bmatrix} \quad (7)$$

where  $\omega = \lambda - \lambda_0$  is a longitude difference from the central meridian of the UTM Zone. For Zone 31  $\lambda_0 = 3^\circ$ .

Using (7) with  $\phi = 50^\circ$ ,  $R_M = 6381822.82$  m and  $\omega = 3^\circ$  (since maximum effects will be on or near zone boundaries); the rounding errors  $\delta \phi = \delta \lambda = 0.00005$  degrees (but expressed in radians) correspond to errors in the  $E, N$  coordinates of  $\delta E = 3.36$  m and  $\delta N = 5.71$  m. These are very similar to the distance errors  $\delta s_m$  and  $\delta s_p$  above.

So we can conclude that rounding the geographical coordinates to the nearest 0.0001 degrees corresponds to UTM Grid coordinates accurate to the nearest 5 metres. For the example that follows all UTM Grid coordinates are shown rounded to the nearest 5 metres.

## Control Points for Conformal Transformation

For the 2D Conformal transformation the following five points were used as control points.

Control Point	British Army Grid Reference	British Army Grid coordinates	Geographical coordinates	UTM Zone 31 Grid coordinates
<b>Woesten</b> Intersection of N8 road and Steenstraat	28.A.6.b.73	$U = 5850$ yards $V = 21650$ yards	$\phi = 50.9007$ deg N $\lambda = 2.7891$ deg E	$E = 485170$ m $N = 5638805$ m
<b>Passchendaale</b> Intersection of N303 road and Zuidstraat	28.D.6.d.32	$U = 23650$ yards $V = 21100$ yards	$\phi = 50.8995$ deg N $\lambda = 3.0206$ deg E	$E = 501450$ m $N = 5638650$ m
<b>Rolleghemcappelle</b> Intersection of Roeselarestraat and Rollegemkapelsestraat	28.F.28.c.84	$U = 33400$ yards $V = 17200$ yards	$\phi = 50.8689$ deg N $\lambda = 3.1490$ deg E	$E = 510485$ m $N = 5635255$ m
<b>Messines</b> Intersection of N314 and N365 roads	28.O.32.d.52	$U = 13750$ yards $V = 5100$ yards	$\phi = 50.7664$ deg N $\lambda = 2.8965$ deg E	$E = 492700$ m $N = 5623855$ m
<b>Reckem</b> Intersection of N366 road and Lauwestraat	28.R.23.c.60	$U = 34300$ yards $V = 7000$ yards	$\phi = 50.7846$ deg N $\lambda = 3.1627$ deg E	$E = 511470$ m $N = 5625885$ m

Table 1. Control Point information for 2D Conformal transformation.

- Note in Table 1:
- (i) British Army Grid coordinates are shown as  $U$  and  $V$  to avoid confusion with the UTM Grid coordinates  $E$  and  $N$ .  $U$  is an east coordinate and  $V$  is a north coordinates.
  - (ii) Geographical coordinates  $\phi, \lambda$  are from Google Maps and rounded to the nearest 0.0001 degrees.
  - (iii) UTM Grid coordinates  $E, N$  have been rounded to the nearest 5 metres.

## Computing the Transformation Parameters: Scale, Rotation and Translations

The transformation parameters (scale  $s$ , rotation  $\theta$ , and translations  $t_E, t_N$ ) are computed in the following sequence.

1. Calculate the coordinates of the centroid in the  $E, N$  and  $U, V$  systems where

$$E_C = \frac{\sum_{k=1}^m E_k}{m}, \quad N_C = \frac{\sum_{k=1}^m N_k}{m}, \quad U_C = \frac{\sum_{k=1}^m U_k}{m}, \quad V_C = \frac{\sum_{k=1}^m V_k}{m} \quad (8)$$

and there are  $m = 5$  control points. Using the values from Table 1 gives

$$\begin{aligned} E_C &= 500255.0000 \text{ m} & U_C &= 22190.0000 \text{ yards} \\ N_C &= 5632490.0000 \text{ m} & V_C &= 14410.0000 \text{ yards} \end{aligned}$$

2. Calculate the centroidal coordinates  $\bar{E}, \bar{N}$  and  $\bar{U}, \bar{V}$  where

$$\begin{aligned} \bar{E}_k &= E_k - E_C & \bar{U}_k &= U_k - U_C \\ \bar{N}_k &= N_k - N_C & \bar{V}_k &= V_k - V_C \end{aligned} \quad (9)$$

Using values from Table 1 and the coordinates of the centroid from above gives

Control Point	$\bar{E}$	$\bar{N}$	$\bar{U}$	$\bar{V}$
Woesten	-15085.000	6315.000	-16340.000	7240.000
Passchendaele	1195.000	6160.000	1460.000	6690.000
Rolleghemcappelle	10230.000	2765.000	11210.000	2790.000
Messines	-7555.000	-8635.000	-8440.000	-9310.000
Reckem	11215.000	-6605.000	12110.000	-7410.000

Table 2. Centroidal coordinates of control points

3. Calculate three sums  $S_1, S_2, S_3$  using the values in Table 2 where

$$\begin{aligned} S_1 &= \sum_{k=1}^m (\bar{U}_k^2 + \bar{V}_k^2) = 859219000.0000 \\ S_2 &= \sum_{k=1}^m (\bar{U}_k \bar{E}_k + \bar{V}_k \bar{N}_k) = 786470000.0000 \\ S_3 &= \sum_{k=1}^m (\bar{V}_k \bar{E}_k - \bar{U}_k \bar{N}_k) = -15140250.0000 \end{aligned} \quad (10)$$

4. Calculate the transformation quantities  $a = s \cos \theta$ ,  $b = s \sin \theta$  where

$$\begin{aligned} a &= \frac{S_2}{S_1} = \frac{786470000.0000}{859219000.0000} = 0.915331248 \\ b &= \frac{S_3}{S_1} = \frac{-15140250.0000}{859219000.0000} = -0.017620944 \end{aligned} \quad (11)$$

5. Calculate the scale  $s$  and rotation  $\theta$

$$\begin{aligned} s &= \sqrt{a^2 + b^2} = 0.915500842 \\ \theta &= \arctan\left(\frac{b}{a}\right) = -1.102859 \text{ degrees} = -1^\circ 06' 10'' \end{aligned} \quad (12)$$

6. Calculate the translations  $t_E, t_N$

$$\begin{aligned} t_E &= E_C - aU_C - bV_C = 480197.717 \text{ m} \\ t_N &= N_C + bU_C - aV_C = 5618909.068 \text{ m} \end{aligned} \quad (13)$$

## Computing the UTM coordinates of other points

With the transformation parameters  $a, b$  and the coordinates of the centroid in both systems  $E_c, N_c$  and  $U_c, V_c$  UTM Grid coordinates of other points may be obtained in the following sequence of steps

1. Convert the British Army Grid Reference (letter/number) to British Army Grid coordinates  $U, V$
2. Convert the  $U, V$  coordinates to centroidal coordinates  $\bar{U}, \bar{V}$  [see equations (9)]
3. Compute the centroidal coordinates  $\bar{E}, \bar{N}$  using equations (21) expressed as  $\bar{E}_k = a\bar{U}_k + b\bar{V}_k$  and  $\bar{N}_k = a\bar{V}_k - b\bar{U}_k$
4. Convert the centroidal coordinates  $\bar{E}, \bar{N}$  to UTM Grid coordinates  $E, N$  [see equations (9)]

## Assessing the accuracy of the transformation

The equations above (see also Appendix B) for the evaluation of the transformation parameters are the result of a least squares estimation process where equations of the form

$$\begin{aligned} v_{E_k} - a\bar{U}_k - b\bar{V}_k &= -\bar{E}_k \\ v_{N_k} - a\bar{V}_k + b\bar{U}_k &= -\bar{N}_k \end{aligned} \quad (14)$$

are written for each  $k = 1, 2, \dots, m$  control point and then ‘solved’ for values of  $a$  and  $b$  that make the sum of the squares of the residuals  $(v_{E_k}, v_{N_k})$  a minimum value. Here the residuals are a measure of inconsistency in the coordinates (UTM and Trench Map) of the control points and a residual is a small correction to an observation such that: observed value + residual = true value.

After  $a$  and  $b$  have been evaluated then the residuals can be obtained from a re-arrangement of (14) and the accuracy of the transformation can be assessed by inspection of the residuals.

Control Point	$v_E$	$v_N$
Woesten	0.912	24.072
Passchendaele	23.500	-10.707
Rolleghemcappelle	-18.299	-13.695
Messines	-6.345	-35.455
Reckem	0.233	35.785

Table 3. Residuals  $v_E, v_N$  (metres) at control points

The residuals in the transformation have the property that

$$\sum_{k=1}^m v_{E_k} = 0 \quad \text{and} \quad \sum_{k=1}^m v_{N_k} = 0 \quad (15)$$

which can be deduced from equations (14), (9) and (8); and confirmed by addition of the values in Table 3.

An assessment of the ‘accuracy’ of the transformation can be made by inspecting the magnitude of the residuals. But what range of numbers would reflect an acceptable result?

Consider the British Army Grid References where the last two numbers represent east and north ‘tenths’ of a Minor-Square’s sides (1 tenth = 50 yards); e.g. Woesten X: 28.A.6.b.73 where ‘73’ represents 7 tenths east and 3 tenths north. Now suppose that the ‘action’ of estimating implies a rounding error of half of the last digit, i.e., half of 1 tenth or 25 yards. We might say that the British Army Grid References (and hence British Army Grid coordinates) are rounded to the nearest 25 yards. And 1 yard = 0.9144 metres so 25 yards = 22.860 metres.

The UTM Grid coordinates of the control points have been derived from latitudes and longitudes obtained from Google Maps and rounded to the nearest 0.0001 degrees. This rounding corresponds to UTM Grid coordinates accurate to the nearest 5 metres (see the analysis above)

Bearing in mind the assumed accuracy of the coordinates of the control points in the transformation (British Army 20–25 metres; UTM approx. 5 metres) it would appear that the residuals at the control points are not excessively large and that the transformation parameters (scale, rotation, translations) are acceptable. Perhaps if residuals at a point were two or three times larger than the assumed accuracy of the coordinates then some suspicion might be attached to those coordinates and map references and Google Maps values checked.

The scale computed from the transformation is 0.915500842 [see equations (12)] and the conversion factor from yards to metres is 0.9144 (yards  $\times$  0.9144 = metres). Converting 1000 yards to metres using the computed scale gives 915.501 m. The correct value is 914.400 m, so the error, expressed as  $x$  per 1000 yards, from using the computed scale is obtained from  $\frac{915.501 - 914.400}{914.400} = \frac{x}{1000}$  and  $x \approx 1.2$ . This is an error of approximately 1 metre per kilometre. This scale error would not be detected on any plotting on a map at 1:40,000 scale.

The rotation computed from the transformation is  $-1^\circ 06'10''$  [see equations (12)] and the negative sign indicates that this is a clockwise angle (in the mathematics of the Conformal transformation angles are considered as positive anti-clockwise). This means that the British Army Grid ( $U, V$  coordinates) must be rotated clockwise by  $1^\circ 06'10''$  to coincide with the UTM Grid ( $E, N$  coordinates) and if the north direction of the UTM Grid (or Grid North) is 'up the page' then the north direction of the British Army Grid is pointing slightly to the west and the angle between them is  $1^\circ 06'10''$ .

The British Army Grid was aligned with the meridian through Brussels, and taking the geographical coordinates of Brussels to be  $\phi = 50^\circ 51'00''$  and  $\lambda = 4^\circ 21'00''$  (<http://en.wikipedia.org/wiki/Brussels>) then the grid convergence<sup>7</sup> at Brussels on a UTM projection (Zone 31,  $\lambda_0 = 3^\circ$ ) is  $\gamma = 1^\circ 02'49''$  and True North<sup>8</sup> is west of Grid North. At the centre of Map 28 the British Army Grid Reference and Grid coordinates are: 28.J.31.c.00 = 18000 yards E. 11000 yards N. And using the computed transformation parameters these grid coordinates can be transformed to UTM Grid coordinates  $E = 496480$  m and  $N = 5629295$  m rounded to the nearest 5 metres. These can be transformed to geographic coordinates  $\phi = 50^\circ 48'55.3''$  and  $\lambda = 2^\circ 57'00.1''$  (rounded to nearest 0.1 seconds of arc) and the grid convergence at the centre of Map 28 is  $\gamma = 0^\circ 02'19''$  and True North is east of Grid North. Combining these two values of grid convergence (at Brussels and the centre of Map 28) gives an angle between the British Army Grid and True North of  $1^\circ 05'08''$  where the north direction of the British Army Grid is west of True North. This value is reasonably close to a value of  $1^\circ 04'$  published by the British Army in 1916 (British General Staff 1916). Also, applying the grid convergence at the centre of Map 28 to the rotation computed from the transformation gives an angle between the British Army Grid and True North of  $1^\circ 06'10'' + 0^\circ 02'19'' = 1^\circ 08'29''$ .

These three values:  $1^\circ 05'08''$ ,  $1^\circ 04'$ ,  $1^\circ 08'29''$  are all reasonably close considering that an angular difference of  $05'$  in 20 km is equivalent to a perpendicular distance of 29 metres (remembering that Map 28 is 32 km east-west and 20 km north-south) and it would seem that the rotation computed from the transformation between the British Army Grid and the UTM Grid is acceptable.

## Conclusion

This paper has given a description of the connection between British Army Grid References (letter/number) and British Army Grid Coordinates ( $E, N$ ). And then shown, using Google Maps and Internet software, how a set of control points having both British Army Grid coordinates and UTM Grid coordinates can be used to compute the parameters of a 2D Conformal transformation. These parameters can then be used to convert British Army Grid coordinates of other points directly to UTM Grid coordinates. A worked example of a transformation has been provided with an Appendix explaining the transformation and associated equations. These may be of use to an interested reader who wishes to develop their own software (Excel).

<sup>7</sup> Grid convergence  $\gamma$  at a point on a map projection is the angle between True North and Grid North.

<sup>8</sup> True North is the direction of the meridian at a point.

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<https://ia601406.us.archive.org/1/items/mapsandartiller00stafgoog/mapsandartiller00stafgoog.pdf> [accessed 11-Jan-2015]
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- Karney, C.F.F., 2011, 'Transverse Mercator with an accuracy of a few nanometers', *Journal of Geodesy*, Vol. 85, No. 8, pp. 475-485.
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<http://journals.sfu.ca/archivar/index.php/archivaria/download/.../12436> [accessed 11-Jan-2015]
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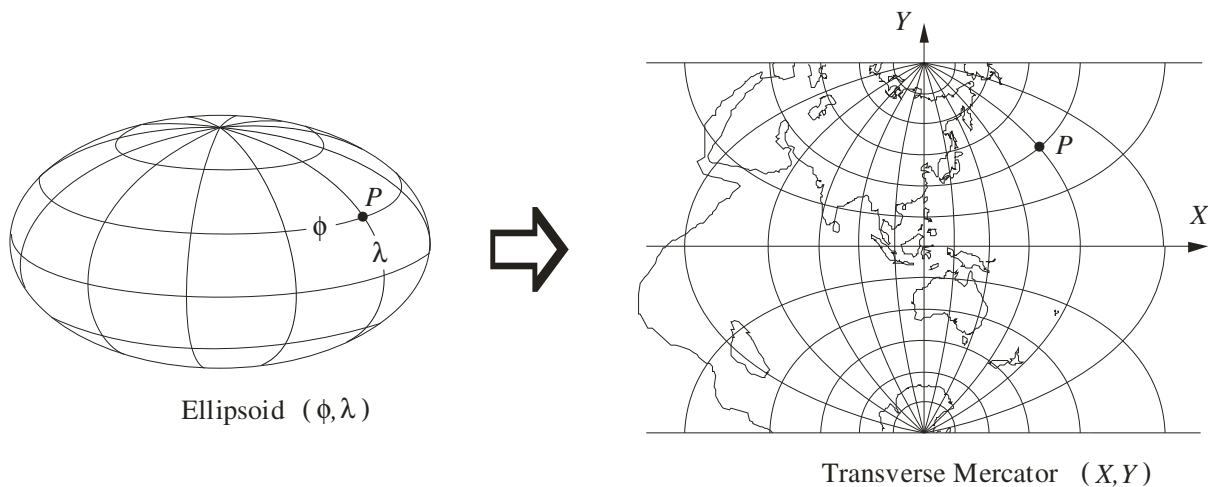
## Sources of Information

- GWD, Great War Digital Ltd, P.O.Box 272, Orpington, BR6 0GX, Kent, England  
<http://www.greatwardigital.com>
- McMaster, McMaster University Library, McMaster University, Ontario, Canada  
[http://library.mcmaster.ca/maps/ww1/WW1\\_Maps.htm](http://library.mcmaster.ca/maps/ww1/WW1_Maps.htm)  
<http://digitalarchive.mcmaster.ca/islandora/object/macrepo%3A4193/-/collection> [map 28 edition 4, 1918/01]



## APPENDIX A

### Transverse Mercator Projection (a diagrammatic explanation)



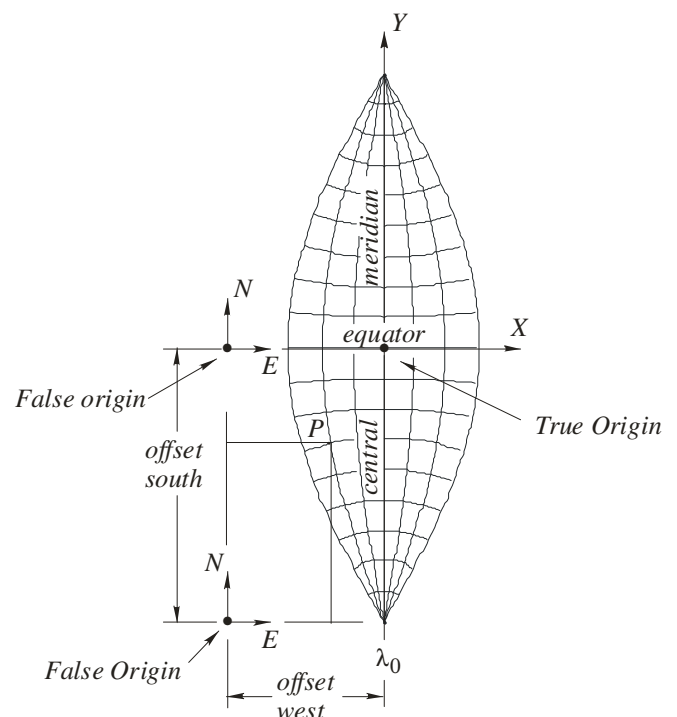
The map projection used for mapping and coordination in Australia and many other parts of the world is the Transverse Mercator (TM). This is a conformal projection of the ellipsoid (the earth reference surface – a slightly squashed sphere) onto a plane (the map projection). Conformal projections preserve shapes of very small regions and the graticule (the arrangement of meridians and parallels) is an orthogonal network. The equations for the transformation  $\phi, \lambda \Leftrightarrow X, Y$  were originally developed in 1822 by the mathematician C.F. Gauss (1777–1855) and L. Krueger published extensive studies including suitable formulae for computation on the projection in 1922. The Transverse Mercator projection is sometimes called the Gauss Conformal or the Gauss-Krueger projection and was adopted with certain restrictions, by the U.S. Army in 1947 as the basis for military mapping. This special case of the TM projection was called the Universal Transverse Mercator (UTM) projection

The TM projection has two straight lines; the central meridian and the equator, all other meridians and parallels are projected as complex curves that intersect everywhere at right angles. The network of meridians and parallels on the projection is known as the graticule (see above right and right).

The scale factor  $m$  is a ratio and  $m = ds/dS$  where  $ds$  is a differentially small distance on the ellipsoid and  $dS$  is a differentially small distance on the map projection. The scale factor along the central meridian (denoted by  $m_0$ ) is constant but varies at every other point. The further from the central meridian the larger the scale factor and hence larger distortions so for mapping and coordination purposes only narrow zones of longitude are mapped (UTM zones are  $6^\circ$  wide).

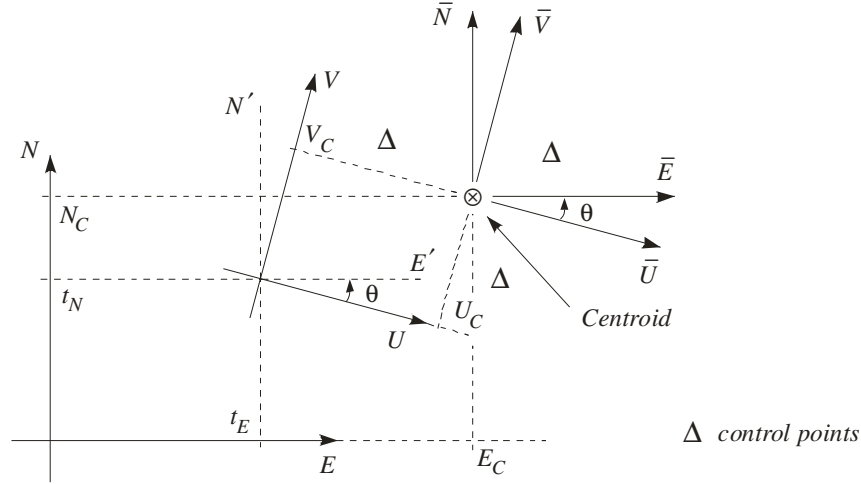
One such zone is shown to the right with a point  $P$  in the southern hemisphere to the west of a central meridian of longitude  $\lambda_0$ . The TM zones used in Australia are  $6^\circ$  wide ( $3^\circ$  either side of the central meridian) and the central meridian has a constant scale factor  $m_0 = 0.9996$ . These TM zones accord with the UTM.

A grid  $E, N$  overlays the graticule and this grid has origins (the False origins) west of the True origin for the northern hemisphere, west and south of the True origin for the southern hemisphere (left above). This grid is often called a TM Grid or UTM Grid.



## APPENDIX B

### 2D Conformal Transformation



In the diagram the  $E, N$  coordinate axes are the axes of the UTM Zone 31 East and North coordinates and the  $U, V$  axes are the axes of the Trench Map East and North coordinates. In this exercise the origin of the  $U, V$  coordinates is the south-west corner of the Trench Map. The  $U, V$  axes are translated ( $t_E, t_N$ ) and rotated ( $\theta$ ) with respect to the  $E, N$  axes, and it is assumed that there is a scale factor  $s$  connecting distances measured in UTM and the Trench Map ‘systems’. The symbol  $\Delta$  denotes a control point that has coordinates in the UTM system and the Trench Map system and the symbol  $\otimes$  denotes the centroid<sup>9</sup> of the control points.  $E_C, N_C$  and  $U_C, V_C$  are the coordinates of the centroid in both systems.

A 2D Conformal Transformation between the two coordinate systems can be expressed as

$$\begin{bmatrix} E_k \\ N_k \end{bmatrix} = s \mathbf{R}_\theta \begin{bmatrix} U_k \\ V_k \end{bmatrix} + \begin{bmatrix} t_E \\ t_N \end{bmatrix} \quad (16)$$

where  $\mathbf{R}_\theta = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$  is a rotation matrix.

In words, equation (16) can be stated as:  $U, V$  coordinates are rotated; scaled; and then translated to  $E, N$  coordinates.

In (16) we may write  $s \mathbf{R}_\theta = \begin{bmatrix} s \cos \theta & s \sin \theta \\ -s \sin \theta & s \cos \theta \end{bmatrix}$ . Letting  $a = s \cos \theta$  and  $b = s \sin \theta$  then scale  $s = \sqrt{a^2 + b^2}$  and

rotation  $\theta = \arctan\left(\frac{b}{a}\right)$  and the transformation can be written as

$$\begin{bmatrix} E_k \\ N_k \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} U_k \\ V_k \end{bmatrix} + \begin{bmatrix} t_E \\ t_N \end{bmatrix} \quad (17)$$

Equations (16) and (17) are different forms of a 2D Conformal Transformation and there are four unknown parameters:  $a$  and  $b$  (from which scale  $s$  and rotation  $\theta$  can be determined); and the translations  $t_E, t_N$ .

<sup>9</sup> The centroid of a series of points in a plane is the average of the coordinates of those points. For three points that form a plane triangle, the centroid lies at the centre of the triangle. A centroid can be thought of as a centre of mass.

Pairs of coordinates for the  $k = 1, 2, \dots, m$  control points yield pairs of equations of the form

$$\begin{aligned} E_k &= aU_k + bV_k + t_E \\ N_k &= -bU_k + aV_k + t_N \end{aligned} \quad (18)$$

and two control points are the minimum required for a solution for the four unknown parameters. In practice three or more control points are used and a mathematical estimation process known as least squares is used to determine the most probable values of the parameters.

The solution can be simplified by reducing the unknown parameters to two by moving the coordinate axes to the centroid of the  $m$  control points where the coordinates of the centroid are

$$E_C = \frac{\sum_{k=1}^m E_k}{m}, \quad N_C = \frac{\sum_{k=1}^m N_k}{m}, \quad U_C = \frac{\sum_{k=1}^m U_k}{m}, \quad V_C = \frac{\sum_{k=1}^m V_k}{m} \quad (19)$$

and the centroidal coordinates are

$$\begin{aligned} \bar{E}_k &= E_k - E_C, & \bar{U}_k &= U_k - U_C \\ \bar{N}_k &= N_k - N_C, & \bar{V}_k &= V_k - V_C \end{aligned} \quad (20)$$

The transformation in centroidal coordinates becomes

$$\begin{bmatrix} \bar{E}_k \\ \bar{N}_k \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} \bar{U}_k \\ \bar{V}_k \end{bmatrix} \quad (21)$$

and there are only two unknown parameters  $a$  and  $b$ .

To solve for these two parameters using the  $m$  control points an observation equation of the form

$$\begin{bmatrix} \bar{E}_k \\ \bar{N}_k \end{bmatrix} + \begin{bmatrix} v_{E_k} \\ v_{N_k} \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} \bar{U}_k \\ \bar{V}_k \end{bmatrix} \quad (22)$$

can be used where residuals<sup>10</sup>  $v_{E_k}, v_{N_k}$  are simply added to the left-hand-side of (21) to account for the assumed inconsistency in the coordinate pairs of the control points. We can think of the residuals as consisting of one part associated with the UTM coordinates  $E, N$  and the other part associated with the Trench Map coordinates  $U, V$ .

Equation (22) can be re-arranged as

$$\begin{aligned} v_{E_k} - a\bar{U}_k - b\bar{V}_k &= -\bar{E}_k \\ v_{N_k} - a\bar{V}_k + b\bar{U}_k &= -\bar{N}_k \end{aligned} \quad (23)$$

For the  $m$  control points the observation equations can be expressed in partitioned matrix form as

$$\begin{bmatrix} v_{E_1} \\ v_{E_2} \\ v_{E_3} \\ \vdots \\ v_{E_m} \\ v_{N_1} \\ v_{N_2} \\ v_{N_3} \\ \vdots \\ v_{N_m} \end{bmatrix} + \begin{bmatrix} -\bar{U}_1 & -\bar{V}_1 \\ -\bar{U}_2 & -\bar{V}_2 \\ -\bar{U}_3 & -\bar{V}_3 \\ \vdots & \vdots \\ -\bar{U}_m & -\bar{V}_m \\ \bar{V}_1 & \bar{U}_1 \\ \bar{V}_2 & \bar{U}_2 \\ \bar{V}_3 & \bar{U}_3 \\ \vdots & \vdots \\ \bar{V}_m & \bar{U}_m \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\bar{E}_1 \\ -\bar{E}_2 \\ -\bar{E}_3 \\ \vdots \\ -\bar{E}_m \\ -\bar{N}_1 \\ -\bar{N}_2 \\ -\bar{N}_3 \\ \vdots \\ -\bar{N}_m \end{bmatrix} \quad (24)$$

<sup>10</sup> A residual is a small correction to an observation such that observed value + residual = true value.

This is in a standard matrix form  $\mathbf{v} + \mathbf{B}\mathbf{x} = \mathbf{f}$  where  $\mathbf{v}$  is the vector of residuals,  $\mathbf{B}$  is the coefficient matrix containing the  $\bar{U}, \bar{V}$  coordinates,  $\mathbf{x}$  is the vector of unknowns  $(a, b)$  and  $\mathbf{f}$  is the vector of numeric terms  $(\bar{E}, \bar{N})$  and the least squares solution (the solution that minimizes the sum of the squares of the residuals) for  $\mathbf{x}$  is given by

$$\mathbf{x} = \mathbf{N}^{-1}\mathbf{t} \quad (25)$$

where

$$\mathbf{N} = \mathbf{B}^T \mathbf{B} = \begin{bmatrix} \sum_{k=1}^m (\bar{U}_k^2 + \bar{V}_k^2) & 0 \\ 0 & \sum_{k=1}^m (\bar{U}_k^2 + \bar{V}_k^2) \end{bmatrix} \quad (26)$$

$$\mathbf{t} = \mathbf{B}^T \mathbf{f} = \begin{bmatrix} \sum_{k=1}^m (\bar{U}_k \bar{E}_k + \bar{V}_k \bar{N}_k) \\ \sum_{k=1}^m (\bar{V}_k \bar{E}_k - \bar{U}_k \bar{N}_k) \end{bmatrix} \quad (27)$$

Because  $\mathbf{N}$  is diagonal the solution for  $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$  is simply

$$a = \frac{\sum_{k=1}^m (\bar{U}_k \bar{E}_k + \bar{V}_k \bar{N}_k)}{\sum_{k=1}^m (\bar{U}_k^2 + \bar{V}_k^2)}; \quad b = \frac{\sum_{k=1}^m (\bar{V}_k \bar{E}_k - \bar{U}_k \bar{N}_k)}{\sum_{k=1}^m (\bar{U}_k^2 + \bar{V}_k^2)} \quad (28)$$

After evaluating  $a, b$  the translations  $t_E, t_N$  are obtained by re-arranging (18) and replacing  $E_k, N_k$  with  $E_C, N_C$  and  $U_k, V_k$  with  $U_C, V_C$  and

$$\begin{aligned} t_E &= E_C - aU_C - bV_C \\ t_N &= N_C + bU_C - aV_C \end{aligned} \quad (29)$$

The scale  $s$  and rotation  $\theta$  are given by

$$s = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \arctan\left(\frac{b}{a}\right)$$

The residuals at the  $m$  control points can be obtained from (22).

The least squares solution for the transformation looks formidable, but it really is very simple. The parameters for any 2D Conformal Transformation can be computed using a calculator (or spreadsheet) and this solution depends on forming only three numbers from a system of centroidal coordinates.